KFKI-71-81

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Hungarian Academy of Sciences

CENTRAL RESEARCH INSTITUTE FOR PHYSICS

BUDAPEST

2017

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ABSTRACT

The stationary states of electrovac fields for which the geodesic eigenrays of both the Maxwell- and the gravitational field coincide are investigated. The fact that the Kerr-Newman solutions belong to this class lends physical interest to the fields considered here. Particular attention is devoted to fields with shearing eigenrays since principal null congruences do <u>not</u> coincide with eigenrays in this case, and so earlier approaches to the problem fail. By the generalization of a theorem on the corresponding vacuum case, it is proven that no "spherical" solutions exist in the shearing class, with the exception of the fields with $\gamma^2 \equiv |G_0|^2 - |H_0|^2 = 0$. The metrics with $\gamma \neq 0$, admitted by the theorem, can either be calculated from the corresponding vacuum solutions by a relatively simple procedure, or, if not, we list them explicitly in this paper.

РЕЗЮМЕ

Изучены стационарные состояния электровакуумных полей, для которых геодезические собственные лучи поля Максвелла и гравитационного поля совпадают друг с другом. Решения Керра-Ньюмана принадлежат к этому классу. Подробно рассматривается случай, в котором собственные лучи обладают сдвигом и $\gamma^2 \equiv |G_0|^2 - |H_0|^2 = 0$.

KIVONAT

Az elektrovákuum-terek stacionárius állapotait vizsgáljuk, amelyekre a gravitációs tér és a Maxwell-tér geodetikus sajátsugarai egybeesnek. A Kerr-Newman megoldások ebbe az osztályba tartoznak. Részletesen azt az esetet tárgyaljuk, amelyre a sajátsugaraknak nyirásuk van, és $\gamma^2 \equiv |G_0|^2 - |H_0|^2 = 0$.

1. INTRODUCTION

In preceding papers, a new approach to stationary states in general relativity has been developed[1,2]. The central idea of the method is the use of a congruence of curves, called eigenrays, which is uniquely determined by the gravitational field. Several unknown features of stationary fields have been discovered by investigating the geometry of eigenrays. Thus, an appropriate condition has been found, which excels the Kerr field among the solutions of the stationary field equations[2]; a theorem on space-times with shearing geodesic eigenrays was proven and new stationary vacuum metrics were constructed in explicit form[3].

This paper is devoted to the investigation of eigenrays in stationary electrovac fields, consisting of interacting /but otherwise source-free/ electromagnetic and gravitational fields[4]. The properties of the electrovacuum have been studied by a number of authors[5,6,7]. In particular, several forms of the stationary electrovac field equations have been found, each having its advantage. The problem, however, is yet far from complete understanding. One stationary electrovac space of physical importance, namely the Newman _et al_. solution[15] is already known. This solution has a straightforward interpretation in the framework of the present approach, as will be shown in Sec.2. The eigencongruence of the Newman _et al_. fields is geodesic and shearfree. In Sec. 3. we shall release the shearfree condition. In the absence of electromagnetism, the fields with shearing geodesic eigenrays had been found to belong to the cylindrical class[3], the physical significance of which is at least dubious. There is, however, no _a priori_ reason for assuming that the same restriction must hold for electrovac spaces. Therefore we have investigated the question whether spherical _electrovac_ fields with shearing geodesic eigenrays are admitted by the field equations.

We would like to stress that the fields considered in this paper do _not_ constitute the electrovac generalizations of the Newman-Tamburino vacuum metrics[13], for which one of the principal congruences of the curvature tensor is assumed to vanish. /We do not know either, if such a generalization is possible./ Rather, our solutions are analogous to the vacuum space-times with shearing geodesic eigenrays[3]. As shown in Ref. 2., the eigenrays of empty space-times do not lie in any of the principal null directions of the curvature, if there is a shear present. This obtains for electrovac fields as well, as will be discussed in Sec. 4. It will be shown, that, even in the presence of electromagnetism there are no solutions with the desired properties and with $\gamma^2 \equiv |G_0|^2 - |H_0|^2 \neq 0$. The exceptional class, $\gamma^2 = 0$, will not be considered here since it is part of a wider family of electrovac fields available by a different method[8]. All /non-spherical, $\gamma^2 \neq 0$/ electrovac metrics admitted by the restrictions of Sec. 4. will be given in Sec. 5. We conclude this paper with some general speculations about the implications of our results for continuum theories.

2. FIELD EQUATIONS

In general relativity, a system of interacting fields is said to be <u>stationary</u> if an everywhere timelike Killing vector field exists in the space-time, along which the Lie-derivatives of all field quantities vanish[9]. An appropriate choice of the coordinate system can be made for such systems by setting the $x^0 = t$ coordinate lines tangent to the Killing field. This condition is still preserved by the following "permissible" coordinate transformations[10]:

$$t' = t + u(x^i), \qquad /2.1a/$$

$$x^{i'} = x^{i'}(x^k). \qquad /2.1b/$$

The Lie-derivatives along the Killing field reduce in this coordinate system to partial derivatives with respect to the time coordinate t, thus the field quantities are independent of t.

Electrovac fields are governed by the coupled Einstein-Maxwell equations. A first order form of the stationary field equations, appropriate for our purposes, results by generalizing the formalism of Ref. 1, where the axisymmetric subclass has been considered. A straightforward but lengthy computation shows that the field equations /37/ and /39/ of Ref. 1 formally remain valid for arbitrary stationary electrovac fields. These equations refer to the metric g_{ik} of the 3-space /named V_3/ of the Killing trajectories. The 3-space is related to ordinary space-time by the covariant decomposition of the line element

$$d\tilde{s}^2 = - f^{-1} ds^2 + f \, (dt + \omega_i \, dx^i)^2 \qquad /2.2/$$

Here $d\tilde{s}$ is the line element in space-time /quantities referring to the space-time and its metric $\tilde{g}_{\mu\nu}$ are tilded here/ $ds^2 = g_{ik} dx^i dx^k$ serves as the definition of V_3; $f = \xi_\mu \xi^\mu$ is the norm of the Killing field ξ^μ and ω_i is a 3-vector with respect to transformations /2.1b/ and transforms under /2.1a/ as follows;

$$\omega_i{}' = \omega_i - u_{,i} \,. \qquad /2.3/$$

Complex 3-vectors are introduced in terms of the electromagnetic vector potential A_μ and the metric field quantities[10/]:

$$\underline{G} = \frac{\nabla\varepsilon + 2\bar{\Phi}\nabla\phi}{2f} \qquad /2.4a/$$

$$\underline{H} = \frac{\nabla\phi}{\sqrt{f}} \qquad /2.4b/$$

with

$$\phi \doteq A_o + iA' \qquad /2.5a/$$

$$\varepsilon = f - |\phi|^2 + i\varphi \qquad /2.5b/$$

and

$$A'_{,i} = \epsilon_{ijk}\,(A^{j,k} - \omega^j\, A_o{}^{;k})f\sqrt{g} \qquad /2.5c/$$

$$\varphi_{,i} = \epsilon_{ijk}\,\omega^{j;k} f^2\sqrt{g} + 2\ \mathrm{Im}\,(\phi\bar{\Phi}_{;i}) \qquad /2.5d/$$

The field equations provide the integrability conditions for the real scalar functions φ and A' further the equations of motion of the vector fields $\underline{G}$ and $\underline{H}$.

$$(\nabla - \underline{G})\,\underline{G} = \bar{\underline{H}}\underline{H} - \bar{\underline{G}}\underline{G} \qquad /2.6a/$$

$$\nabla \times \underline{G} = \bar{\underline{H}} \times \underline{H} - \bar{\underline{G}} \times \underline{G} \qquad /2.6b/$$

$$(\nabla - \underline{G})\,\underline{H} = \frac{1}{2}\,(\underline{G} - \bar{\underline{G}})\,\underline{H} \qquad /2.6c/$$

$$\nabla \times \underline{H} = -\frac{1}{2}\,(\underline{G} + \bar{\underline{G}}) \times \underline{H} \qquad /2.6d/$$

$$R_{ik} + G_i\bar{G}_k + \bar{G}_i G_k - H_i\bar{H}_k - \bar{H}_i H_k = 0. \qquad /2.6e/$$

Here $R_{ik} = R_{iv}{}^{v}{}_{k}$ is the Ricci tensor of V_3 and the notation

$$(\underline{A} \times \underline{B})_i \stackrel{d}{=} \epsilon_{ijk}\, A^j\, B^k\, \sqrt{g} \qquad /2.7/$$

is used.

It should be noted that a second order form of the field equations can be found which is symmetric in the complex scalars ξ and q given by

$$\varepsilon = \frac{\xi - 1}{\xi + 1} \qquad \phi = \frac{q}{\xi + 1} . \qquad /2.8/$$

We have, using /2.8/,

$$(\xi\bar{\xi} + q\bar{q} - 1)\ \Delta\xi = 2\left[\bar{\xi}(\nabla\xi)^2 + \bar{q}\nabla q\nabla\xi\right] \qquad /2.9a/$$

$$(\xi\bar{\xi} + q\bar{q} - 1)\ \Delta q = 2\left[\bar{q}(\nabla q)^2 + \bar{\xi}\nabla\xi\nabla q\right] \qquad /2.9b/$$

$$R_{ij} = -2\left(\xi\bar{\xi} + q\bar{q} - 1\right)^{-2} \mathrm{Re}\Big[\left(1 - q\bar{q}\right)\xi_{,i}\bar{\xi}_{,j} + $$
$$+ \left(1 - \xi\bar{\xi}\right)q_{,i}\ \bar{q}_{,j} + \bar{\xi}q\ \xi_{,i}\ \bar{q}_j + \xi\bar{q}\ \bar{\xi}_{,j}\ q_{,i}\Big] . \qquad /2.9c/$$

In the absence of electromagnetism, $q = 0$ and field equations /2.9/ reduce to the generalization of the Ernst form of vacuum equations:[9]

$$(\xi\bar{\xi} - 1)\Delta\xi = 2\bar{\xi}(\nabla\xi)^2 \qquad /2.10a/$$

$$R_{ij} = -\frac{\xi_{,i}\bar{\xi}_{,j} + \bar{\xi}_{,i}\xi_{,j}}{(\xi\bar{\xi}-1)^2} . \qquad /2.10b/$$

The $q = q^o$ = const. assumption, on the other hand, leads to the vacuum equations provided $|q^o| < 1$. In this case, the substitution

$$\xi' = \frac{\xi}{\sqrt{1-q^o\bar{q}^o}} \qquad /2.11/$$

is to be performed. Solutions with $|q^o| \geq 1$, however, cannot be obtained by this procedure since /2.11/ then does not yield the vacuum equations. This restriction, as far as we know, has not been observed heretofore, even for the axisymmetric subclass.

The correspondence between the vacuum problem /2.10/ and the electrovac one with $q = q^o$, $|q^o| < 1$ reveals an easy way of construction of electrovac metrics from a given vacuum solutions. The resulting field will be named the <u>electrovac counterpart</u> of the corresponding vacuum solution. From our point of view, the essential point in the construction is that the electrovac counterpart inherits the 3-metric g_{ik} of the vacuum field. We can say that the geometry of V_3 is left unaffected when constructing the electrovac counterpart.

3. THE COMPLEX TRIAD FORMALISM

A system of normalized basic vectors, called a triad, is used in the 3-space V_3 to form scalar /triad/ components of all field quantities[2]. The elements of the triad are denoted $\underline{\ell}$, $\underline{m}$, $\underline{\bar{m}}$ with $\underline{\ell}$ real vector and $\underline{\bar{m}}$ being the complex conjugate to $\underline{m}$. The normalization is $\underline{\ell}\underline{\ell} = \underline{m}\,\underline{\bar{m}} = 1$, and all other scalar products between basic vectors vanish. Four complex scalars $/\rho, \sigma, \kappa, \tau/$ and an imaginary one $/\epsilon/$ are introduced according to

$$\kappa = - \ell_{i;j}\, m^i \ell^j \qquad \rho = - \ell_{i;j} m^i \bar{m}^j$$

$$\epsilon = m_{i;j} \bar{m}^i \ell^j \qquad \sigma = - \ell_{i;j}\, m^i m^j$$

$$\tau = m_{i;j}\, \bar{m}^i\, \bar{m}^j . \tag{3.1}$$

These complex quantities can either be regarded as particular SU/2/ spin coefficients[2] or, equivalently, as complex Ricci-rotation coefficients. They suitably characterize the geometry of the congruence of curves in V_3 defined by tangent ℓ^i. The condition $\kappa = 0$ means that the curves are geodesics. Then the divergence, rotation and complex shear of the congruence is given by $\mathrm{Re}\rho$, $\mathrm{Im}\rho$ and σ, respectively[2].

We introduce, in addition, the differential operators

$$D = \ell^i \partial_i, \qquad \delta = m^i \partial_i, \qquad \bar{\delta} = = \bar{m}^i \partial_i \tag{3.2}$$

with commutation properties

$$(D\delta - \delta D) = (\bar{\rho} + \epsilon)\delta + \sigma\bar{\delta} + \kappa D \tag{3.3a}$$

$$(\delta\bar{\delta} - \bar{\delta}\delta) = \bar{\tau}\bar{\delta} - \tau\delta + (\bar{\rho} - \rho)D . \tag{3.3b}$$

By taking triad projections of Eq.s /2.9/, we obtain the invariant first-order form of the stationary electrovac field equations. We remind the reader that there is still a complete freedom in the orientation of triad. Since the triad equations in the generic case are somewhat unwieldy, we put off listing them explicitly until certain restrictions on the triad and field variables are made. It will prove useful to take the basic vector ℓ^i tangent to the eigenray congruence of the graviational field. The definition of eigenrays is, in fact, most conveniently given in the triad notation. It is always possible to set

$$G_i m^i = 0 . \tag{3.4}$$

This condition is apparently invariant against rotations of the triad around the axis $\underline{\ell}$:

$$\underline{\ell}' = \underline{\ell}$$

$$\underline{m}' = e^{iC}\underline{m}. \qquad /3.5/$$

Here C is an arbitrary real function of the coordinates. The direction of $\underline{\ell}$, on the other hand, is uniquely fixed by /3.4/ given the /nonzero/ vector $\underline{G}$. The curves with the unit tangent vector $\underline{\ell}$ determined by /3.4/ are just the eigenrays of the gravitational field[11].

We have now, in addition to $\underline{G}$, the complex 3-vector $\underline{H}$ representing the Maxwell field. So we can analogously define the eigenray congruence of the Maxwell field by demanding

$$H_i m^i = 0. \qquad /3.6/$$

A particular class of the space-times is for which the eigenrays of the gravitational and Maxwell fields coincide. For such fields we can choose a triad such that /3.4/ and /3.6/ hold simultaneously. As is easily verified, a necessary and sufficient condition for the existence of common eigenrays is

$$(\underline{G} \times \underline{H})^2 = 0. \qquad /3.7/$$

An interesting class of electrovac fields satisfying /3.7/ is that of the electrovac counterparts of vacuum solutions. From definitions /2.4/ we see that $\underline{G}$ and $\underline{H}$ are then parallel vectors in the sense that $\underline{G} \times \underline{H} = 0$ holds. Hence it follows that the important Newman et al. solution, being the electrovac counterpart of the Kerr field[5], possesses common eigenrays. As the Kerr field is characterized by shear-free geodesic eigenrays[2] and since the geometry of V_3 is unaffected when constructing the electrovac counterpart, the common eigenrays of the Newman et al. metric must also be geodesic and non-shearing. In the shear-free case the eigenrays lie in one of the principal directions of the four-curvature, such that the Newman-Penrose equations[12] are soluble without any ad hoc assumption. Hence, we will not consider here this class; instead we concentrate on the shearing fields for which the eigenrays do not coincide with the principal directions, yet the triad equations can be treated without any approximation. It should be pointed out that the corresponding vacuum problem has been studied by Kóta and Perjés[3].

4. FIELDS WITH SHEARING GEODESIC EIGENRAYS

The triad vector $\underline{\ell}$ is taken tangent to the eigenrays, therefore

$$G_+ = H_+ = \kappa = 0. \qquad /4.1/$$

Triad rotations /3.6/ are used to make $\epsilon = 0$ and then remains the freedom

$$\underline{\ell}' = \underline{\ell}$$
$$\underline{m}' = e^{iC^o}\underline{m} \qquad /4.2/$$

with $D\,C^o = 0$. To fix the triad completely, $\sigma = \bar{\sigma}$ is set. The field equations can be written

$$D\sigma = (\rho + \bar{\rho})\sigma \qquad /4.3a/$$

$$D\rho = \rho^2 + \sigma^2 + \gamma^2 \qquad /4.3b/$$

$$DG_o = (2\rho + G_o)\,G_o - \gamma^2 \qquad /4.3c/$$

$$DH_o = (2\rho + \tfrac{3}{2}\,G_o - \tfrac{1}{2}\,\bar{G}_o)H_o \qquad /4.3d/$$

$$D\tau = \rho\tau - \sigma\bar{\tau} + \bar{G}_o\,G_- - \bar{H}_o\,H_- \qquad /4.3e/$$

$$DG_- - \bar{\delta}G_o = \rho G_- - \bar{G}_o\,G_- + \bar{H}_o\,H_- \qquad /4.3f/$$

$$DH_- - \bar{\delta}H_o = \rho H_- - \tfrac{1}{2}\,(G_o + \bar{G}_o)H_- + \tfrac{1}{2}\,G_-\,H_o \qquad /4.3g/$$

$$\delta\rho - \bar{\delta}\sigma = -2\sigma\tau + \bar{G}_+G_o - \bar{H}_+H_o \qquad /4.3h/$$

$$\delta\tau + \bar{\delta}\bar{\tau} = 2\tau\bar{\tau} + \rho\bar{\rho} - \sigma^2 - \gamma^2 + G_-\,\bar{G}_+ - H_-\bar{H}_+ \qquad /4.3i/$$

$$-\,\delta G_o = \sigma G_- + \bar{G}_+\,G_o - \bar{H}_+\,H_o \qquad /4.3j/$$

$$-\,\delta H_o = \sigma H_- + \tfrac{1}{2}\,\bar{G}_+H_o \qquad /4.3k/$$

$$-\,\delta G_- = (\rho - \bar{\rho})\,G_o - \bar{\tau}G_- + \bar{G}_+\,G_- - \bar{H}_+\,H_- \qquad /4.3l/$$

$$-\,\delta H_- = (\rho - \bar{\rho})\,H_o - \bar{\tau}H_- + \tfrac{1}{2}\,\bar{G}_+\,H_-\,. \qquad /4.3m/$$

In /4.3/ we have introduced the notation $\gamma^2 = G_o\bar{G}_o - H_o\bar{H}_o$.

We are dealing with the fields for which $\sigma \neq 0$ /the complex shear does not vanish/. Also we will assume that neither H_o nor G_o vanishes. From the assumption $H_o = 0$, namely, it follows by Eq. /4.3k/ that $H_- = 0 = \underline{H}$.

This holds for the vacuum, discussed in Ref. 3, and so will be excluded here. From Eq. /4.3c/ we see that $G_o \neq 0$ as well.

Taking mixed derivatives of field equations /4.3c/ and /4.3j/ and using the commutators /3.3/ to eliminate second derivatives as shown in Ref. 3, we get

$$\bar{\delta} \ln (G_o \sigma) = G_- + 2\tau \qquad /4.4a/$$

further, from Eq.s /4.3d/ and /4.3k/,

$$\bar{\delta} \ln (H_o \sigma) = \tfrac{1}{2} G_- + 2\tau + \frac{G_o H_-}{H_o} . \qquad /4.4b/$$

We still need the propagation properties of γ, which follow simply from the field equations,

$$D\gamma = (\rho + \bar{\rho})\, \gamma \qquad /4.5a/$$

$$\bar{\delta}\gamma = (2\tau - \bar{\delta} \ln \sigma)\gamma^2 - (G_o \bar{G}_+ - H_o \bar{H}_+)\sigma \qquad /4.5b/$$

$\gamma = 0$ means, by /4.5b/ that $G_o \bar{G}_+ - H_o \bar{H}_+ = 0$, hence we have then $G_- \bar{G}_+ - H_- \bar{H}_+ = 0$ also. This amounts to $R_{\underline{mn}} = 0$, that is, to the background V_3 being flat. Therefore the fields with $\gamma = 0$ constitute a subclass of the stationary electrovac fields characterized by a flat V_3, a class which has been discussed elsewhere[8]. In the following, we will be concerned with the field for which $\gamma \neq 0$ holds. We shall show that these fields are subject to severe restrictions which arise from the generalization of a theorem found by Kóta and Perjés[3] for the corresponding vacuum problem.

<u>T h e o r e m</u>. In stationary electrovac fields for which the gravitational and electromagnetic fields have a common geodesic eigencongruence and $\gamma \neq 0$, the eigenrays either can have shear or curl but not both; when a shear is present, then conditions

$$\rho\bar{\rho} - \sigma\bar{\sigma} - \gamma^2 = 0 \qquad /4.6/$$

$$\delta\rho = \bar{\delta}\rho = \delta\sigma = \delta\gamma \qquad /4.7/$$

must hold.

The proof, like in the vacuum case, is based on forming mixed derivatives of appropriately chosen field equations and using commutators /3.3/ such that the second order terms cancel. We assume $\sigma \neq 0$ and begin with the pair of equations /4.5/ to obtain

$$3\gamma^2\bar{\delta}\rho + \left(\gamma^2+\sigma^2\right)\delta\sigma + \left(\gamma^2-\sigma^2\right)\left(\bar{\delta}\bar{\rho} + 2\sigma\bar{\tau}\right) = 0. \qquad /4.8/$$

This, apart from the inclusion of electromagnetism into the definition of γ, is just Eq. /13/ of reference 3. Hereupon, we can proceed along the lines of that paper until we get $\delta\sigma = \delta\gamma = \bar{\delta}\rho = \delta\rho$. Next we consider the only nonvanishing component of the vector product $\underline{G} \times \underline{H}$, namely

$$X \stackrel{d}{=} G_o H_- - H_o G_- . \qquad /4.9/$$

From the field equations we get

$$D X = \frac{3}{2}\left(2\rho + G_o - \bar{G}_o\right)X \qquad /4.10a/$$

$$\delta X = \left(\bar{\tau} - \frac{3}{2}\bar{G}_+\right) X \qquad /4.10b/$$

and, multiplying both sides of /4.3h/ by their complex conjugates,

$$X\bar{X} = \gamma^2\left(4\tau\bar{\tau} - G_-\bar{G}_+ + H_-\bar{H}_+\right). \qquad /4.11/$$

Here we have made use of equation $\left(\gamma^2 - \sigma^2\right)\tau = 0$, which is a consequence of /4.8/. Hence we see also that either $\tau = 0$ or $\gamma^2 = \sigma^2$ holds for the fields considered here. In any case, the $\bar{\delta}$ derivative of /4.h/ reads,

$$\bar{\delta}\tau = 3\tau^2 + \frac{1}{2}\left(\rho-\bar{\rho}\right)\sigma . \qquad /4.12/$$

A further action of D operator yields the algebraic relation

$$- 4\tau\bar{\tau} + G_-\bar{G}_+ - H_-\bar{H}_+ + \rho\bar{\rho} - \sigma^2-\gamma^2 - \frac{1}{2}\left(\rho-\bar{\rho}\right)^2 = 0. \qquad /4.13/$$

Under the assumption $X \neq 0$, from the derivatives of Eq.s /4.10/ we obtain an other algebraic constraint on the variables,

$$2\left(\rho\bar{\rho} - \sigma^2 - \gamma^2\right) + \left(\rho - \bar{\rho}\right)^2 = 0. \qquad /4.14/$$

By equations /4.11/, /4.12/ and /4.13/ we are thus driven to $X = 0$, a contradiction to our assumption. This establishes in an indirect way the important by-product of our proof:

L e m m a. $\underline{G}$ and $\underline{H}$ are parallel vectors and $X = 0$.

All missing parts of the proof of the theorem follow from this Lemma. Namely, from /4.11/

$$4\bar{\tau}\tau - G_{-}\bar{G}_{+} + H_{-}\bar{H}_{+} = 0. \qquad /4.15/$$

Supposing firstly that $\tau = 0$, from /4.3h/ and /4.15/ we get $G_{-} = H_{-} = 0$. According to /4.12/, $\rho - \bar{\rho}$ vanishes and so /4.14/ yields $\rho\bar{\rho} - \sigma^2 - \gamma^2 = 0$ /eq. /4.6//.If $\tau \neq 0$ on then other hand, we have, by /4.13/ and /4.15/: $2\left(\rho\bar{\rho} - \sigma^2 - \gamma^2\right) + \left(\rho - \bar{\rho}\right)^2 = 0$. Applying commutator /3.3b/ on γ and using field equation /4.3b/, we recover $\rho - \bar{\rho} = 0$ and $\rho\bar{\rho} - \sigma^2 - \gamma^2 = 0$, which was to be proven.

An immediate consequence of the above Lemma is that fields with $\gamma^2 > 0$ are the electrovac counterparts of the Kóta-Perjés metrics[3/], therefore they can be calculated easily as shown in Sec. 2. For the statement that $\underline{G}$ and $\underline{H}$ are parallel vectors implies, by definitions /2.4/ and /2.8/ that $q = \text{const}$. According to what has been said in Section 2, this property characterizes the electrovac counterparts of vacuum solutions. What remains to be done is to solve the field equations for the $\gamma^2 < 0$ class, taking into account the restrictions carried by the above theorem. There is no point in going through the details of the integration; since, in particular, the corresponding vacuum procedure has been discussed at length in Ref. 3. So we will contend here with giving the final results of the calculation /see Table I./ There are two types of line elements with $\gamma^2 < 0$, each characterized by the vanishing or nonvanishing of ρ, accordingly. This should be contrasted with the corresponding vacuum solutions of Ref. 3, where it was found that $\rho = 0$ is prohibited by the field equations.

We have summarized our results in Table I. All possible types of the stationary electrovac fields with shearing geodesic eigenrays are listed there, together with either the detailed form of the corresponding line element and electromagnetic potential or a reference to the paper to be consulted with on it.

Table I.

The stationary electrovac fields with shearing geodesic eigenrays

<table>
<tr><td>$\gamma=0$</td><td colspan="2">Special case of the more general class discussed in Ref. 8.</td></tr>
<tr><td>$\gamma^2>0$</td><td colspan="2">These fields are the electrovac counterparts of the Kóta-Perjés solutions[3].</td></tr>
<tr><td rowspan="2">$\gamma^2<0$ and $\tau=0$</td><td>$\rho=0$</td><td>

$$ds^2_{(V_3)} = dr^2 + e^{-2\sigma^o r}\, dx^2 + e^{2\sigma^o r}\, dy^2$$

$$f = \frac{f^o}{\cos z \cos \bar{z}} \quad , \quad \Phi = e^{i\vartheta^o} \sqrt{f^o}\, \mathrm{ch}^{1/2}(2\sigma^o Q)\, \mathrm{tg}\, z$$

$$\underline{\omega} = \left(0,\ 0,\ -\frac{\sigma^o}{2f^o}\, \mathrm{sh}(2\sigma^o Q)x\right) , \qquad x^1 = r,\quad x^2 = x,\quad x^3 = y$$

with $z = \sigma^o(r+iQ)$ and

σ^o, Q, f^o, ϑ^o real constants

</td></tr>
<tr><td>$\rho\neq 0$</td><td>

$$ds^2_{(V_3)} = dr^2 + r^{1-\sigma^o}\, dx^2 + r^{1+\sigma^o}\, dy^2$$

$$f = \frac{f^o}{\left(r^{i\gamma^o}+iQ\right)\left(r^{-i\gamma^o}-iQ\right)} \quad , \quad \Phi = e^{i\vartheta^o} \sqrt{\frac{f^o(1+Q\bar{Q})}{2}}\ \frac{1}{r^{i\gamma^o}+iQ}$$

$$\underline{\omega} = \left(0,\ 0,\ \frac{\gamma^o}{f^o}\,(1 - Q\bar{Q})x\right) , \qquad x^1=r,\ x^2=x,\ x^3=y$$

with Q complex constant, and

σ^o, γ^o, f^o, ϑ^o real constants,

$$\sigma^{o2} - \gamma^{o2} = 1$$

</td></tr>
</table>

5. PHYSICAL INTERPRETATION

After a series of similar experiences[3, 13], it is no more surprising that admission of shear for geodesic eigenrays results in exclusion of the "spherical class" characterized by $\rho\bar{\rho} - \sigma\bar{\sigma} - \gamma^2 \neq 0$ of electrovac fields. Newman and Tamburino[13] have speculated that the corresponding phenomenon shown by the principal congruence of the curvature tensor, may be attributed to the nonexistence of self-acceleration of free particles. We would like to suggest yet another way of reasoning which in turn leads to far-reaching conclusions indeed. It can well be, namely, that in the case when no shear-free light-like geodesic congruence exists in a given space-time, then the abstraction of a physical point looses its "absolute" character; an object may appear for some observers a point and for others, as something of finite dimensions. There is, of course, a possibility that we measure the dimensions of the particle by contact methods. Can this really be done? Certainly not, in the strict sense: an elementary object cannot be compared directly with a measuring rod.

On adopting this view, we conclude that the notion of the point, which is an essential ingredient of any continuum theory, in a generic situation will not represent a reasonable physical entity. If this is the case, non-continuum theories /as, for example, Penrose's twistor formalism[14]/ should be necessitated for description of space-time properties.

6. ACKNOWLEDGMENTS

We would like to thank Drs. F. Károlyházy and J. Kóta for illuminating discussions.

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10. Our conventions of notation: i, j, k,... are space-like tensor indices with values 1,2 and 3; whereas $\underline{m},\underline{n},\underline{p}$, ... are triad labels valued $0, +$ and $-$. We have chosen $c = 1$ and the gravitational constant of Einstein, $k = 2$. Covariant derivatives with respect to the 3x3 metric g_{ik} are denoted by a semicolon in the suffix. ∇ and Δ are the 3-covariant gradient /divergence/ and Laplacian operators, respectively, and partial derivatives are denoted $\gamma_{,i}$, 3-vectors are sometimes written, for example as, $\underline{G}$. e_{ijk} stands for the numerical Levy-Civita symbol.
11. It is perhaps confusing that the definition of $\underline{G}$ involves quantities connected with the Maxwell field. This is, however, merely a formality by means of which the connection between $\underline{G}$ and the invariants ε and Φ is established. It can be shown[8] that $\underline{G}$ is written equivalently as $\underline{G} = \dfrac{\nabla f + if^2 \nabla x \underline{\omega}}{2f}$. This relation no more contains electromagnetic terms and conforms with the definition given in Ref. 2. The eigenrays defined by $\underline{Gm} = 0$ thus are defined as before.
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Kiadja a Központi Fizikai Kutató Intézet
Felelős kiadó: Kiss Dezső, a KFKI Nagyenergiáju
Fizikai Tudományos Tanácsának elnöke
Szakmai lektor: Kóta József
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Készült a KFKI sokszorositó üzemében
Budapest, 1972. január hó